A note on the inviscid Orr–Sommerfeld equation

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The inviscid Orr–Sommerfeld equation for $\phi(y)$ in $y > 0$ subject to a null condition as $y \to \infty$ is attacked by considering separately the intervals $(0, y_1)$ and (y_1, ∞), such that the solution in $(0, y_1)$ can be expanded in powers of the wave-number (following Heisenberg) and the solution of (y_1, ∞) regarded as real and non-singular. Complementary variational principles for the latter solution are determined to bound an appropriate parameter from above and below. It also is shown how the original differential equation may be transformed to a Riccati equation in such a way as to facilitate both the Heisenberg expansion of the solution in $(0, y_1)$ and numerical integration in (y_1, ∞). These methods are applied to a velocity profile that is linear in $(0, y_1)$ and asymptotically logarithmic as $y \to \infty$, and it is found that the mean of the two variational approximations is in excellent agreement with the results of numerical integration of the Riccati equation.

1. Introduction

Investigations of the stability of the parallel shear flow $U(y)$ (Lin 1955) or of the transfer of energy from such a shear flow to a surface wave at $y = 0$ (Brooke Benjamin 1959, 1960; Miles 1957*a*, 1959*a*, *b*, 1962) lead to the following boundary-value problem for the inviscid stream function $\phi(y)\exp[ik(x-ct)]$.

Solve the inviscid Orr–Sommerfeld equation

$$(U-c)(\phi''-k^2\phi)-U''\phi = 0 \tag{1.1}$$

for $\phi(y)$, $y > 0$, subject to the null condition

$$\phi(y) \to 0 \quad (ky \to \infty), \tag{1.2}$$

and determine the complex parameter

$$w = u+iv = [1+(U_0'\phi_0/c\phi_0')]^{-1}, \tag{1.3}$$

where k is a real positive wave-number, and c is a complex velocity having a small, positive-imaginary part c_i.

We use primes to imply differentiation with respect to y, the subscript zero to imply evaluation at $y = 0$ (e.g. $\phi_0' \equiv d\phi/dy$ at $y = 0$), and the subscript c to imply evaluation at the singular point $y = y_c$, defined by

$$U(y_c) = c. \tag{1.4}$$

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We shall assume that $U(y)$ is a monotonically increasing function of y that vanishes at $y = 0$.

The following treatment of this boundary-value problem was developed for a specific, surface-wave generation problem (Miles 1962). It appears likely, however, that the method of attack should be of more general interest, and it is for this reason that the analysis is being published separately.

The principal difficulty in solving the boundary-value problem posed by (1.1) to (1.3) is the singularity at $y = y_c$, where ϕ exhibits the behaviour

$$\phi \to \phi_c\{1 + (U_c''/U_c')(y-y_c)\log[k(y-y_c)] + O[k(y-y_c)]\}. \tag{1.5}$$

The solution for $y \gg y_c$ is well behaved, on the other hand, and its asymptotic behaviour is given by

$$\phi(y) \sim e^{-ky}\left[1 - \frac{1}{2k}\int_\infty^y \frac{U''(y)\,dy}{U(y)-c} + \ldots\right], \tag{1.6}$$

provided that $U''/k^2(U-c)$ vanishes uniformly as $ky \to \infty$.

We shall attack the problem on the hypotheses that (a) there exists a real number ky_1 such that

$$|y_c| < y_1 \ll 1/k \tag{1.7a}$$

and (b)

$$|U_1 - c| \gg c_i, \tag{1.7b}$$

where the subscript 1 implies evaluation at $y = y_1$. We then may consider separately the intervals $(0, y_1)$, near which ϕ is singular according to (1.5), and (y_1, ∞), in which the differential equation (1.1) is regular. We shall designate the solutions in these intervals as the *inner* and *outer* solutions and seek approximations to them that tend to the exact solutions as $ky_1 \to 0$ and $c_i \to 0+$, respectively. Before entering into those considerations, however, we shall find it convenient to transform the dependent variable along lines suggested by Lighthill (1953, 1957).

2. Transformations

Introducing the dependent variables θ and ϖ according to

$$\phi = (U-c)\,\theta(y) = k^{-2}(U-c)^{-1}\varpi'(y) \tag{2.1a,b}$$

and the abbreviation

$$Y(y) = (U-c)^2, \tag{2.2}$$

we may transform the second-order differential equation (1.1) to the pair of first-order differential equations

$$Y\theta' = \varpi \quad \text{and} \quad \varpi' = k^2 Y\theta. \tag{2.3a,b}$$

Eliminating either ϖ or θ between (2.3a,b), we obtain the complementary, Sturm–Liouville equations

$$(Y\theta')' - k^2 Y\theta = 0 \tag{2.4}$$

and

$$(Y^{-1}\varpi')' - k^2 Y^{-1}\varpi = 0. \tag{2.5}$$

We remark that $\theta(y)\exp[ik(x-ct)]$ and $\varpi(y)\exp[ik(x-ct)]$ are proportional to the inclination of the streamlines and the perturbation pressure, respectively, in the inviscid flow field of the physical problem.

We also shall find it convenient to introduce

$$\Omega(y) = -\theta/Y\theta' = -\varpi'/k^2 Y\varpi. \tag{2.6a,b}$$

Substituting (2.6a,b) into (2.4) and (2.5), respectively, we obtain the Riccati equation

$$\Omega' = k^2 Y\Omega^2 - Y^{-1}, \tag{2.7}$$

where

$$\Omega \sim (kY)^{-1} \quad \text{as} \quad ky \to \infty. \tag{2.8}$$

Substituting (2.1a) and (2.6a) into (1.3), we obtain

$$w = 1 + U_0' c\,\Omega_0. \tag{2.9}$$

3. Inner solution

Integrating (2.7) inward from $y = y_1$ along a path indented under $y = y_c$, we obtain

$$\Omega = \Omega_1 - \int_{y_1}^{y} Y^{-1} dy + k^2 \int_{y_1}^{y} Y\Omega^2 dy. \tag{3.1}$$

We may use this result to obtain, by iteration, an expansion of Ω_0 in powers of k^2. This is, in essence, Heisenberg's procedure (Lin 1955, §3.4) for solving the boundary-value problem posed by (1.1)–(1.3). In (what is equivalent to) the conventional application of this procedure, one assumes $U' \equiv 0$ in $y \geqslant y_1$ and takes $\Omega_1 = 1/kY_1$ in accord with (2.8).

Invoking the restriction (1.7a), we shall rest content with the first approximation to (3.1), which yields

$$\Omega_0 = (\Omega_1 + K_1)\,[1 + O(ky_1)^2], \tag{3.2}$$

where

$$K_1 = \int_0^{y_1} (U-c)^{-2}\, dy. \tag{3.3}$$

Integrating (3.3) by parts, we obtain

$$J = K_1 + (U_0' c)^{-1} = -[U_1'(U_1 - c)]^{-1} - \int_0^{y_1} U'^{-2} U''(U-c)^{-1}\, dy. \tag{3.4}$$

Substituting (3.2) and (3.3) into (2.9), we obtain

$$w = U_0' c(\Omega_1 + J)\,[1 + O(ky_1)^2]. \tag{3.5}$$

Invoking the restriction (1.7b), we have the known result (Lin 1955) that the imaginary part of w is derived almost entirely from the integral J according to

$$v = -\pi(U_c'' U_0' c/U_c'^3)\,[1 + O(k\,|y_c|)^2]. \tag{3.6}$$

4. Outer solution

It remains to determine the parameter Ω_1 from the outer solution. We first remark that, in so far as we may approximate $U_1 - c$ as real in $y > y_1$ by virtue of (1.7b), $Y(y)$ is a positive definite function in (y_1, ∞). Applying Sturm's oscillation theorem (Ince 1944) to the differential equations (2.4) and (2.5) and invoking the requirement that each of θ and ϖ vanishes at $y = \infty$, it then follows that neither θ nor ϖ can vanish in (y_1, ∞).† This implies that the function Ω is bounded and non-oscillatory in (y_1, ∞), in consequence of which its solution by numerical integration is straightforward. In carrying out this integration it generally would appear to be expedient to introduce Y, rather than y, as the independent variable and to integrate inward from Y_2, $\Omega_2 = 1/kY_2$ to Y_1, Ω_1, where $ky_2 \gg 1$.

We also may obtain variational approximations to Ω_1 that bound it from above and below. Multiplying the differential equation (2.4) through by either θ or its complex conjugate θ^*, integrating by parts over (y_1, ∞), invoking the null condition at $y = \infty$, and solving the resulting expressions for Ω_1, we obtain

$$\Omega_1^{-1} = \theta_1^{-2} \int_{y_1}^{\infty} (\theta'^2 + k^2\theta^2)\, Y\, dy \tag{4.1a}$$

$$= |\theta_1|^{-2} \int_{y_1}^{\infty} (|\theta'|^2 + k^2\, |\theta|^2)\, Y\, dy. \tag{4.1b}$$

Operating similarly on (2.5), we obtain

$$\Omega_1 = (k\varpi_1)^{-2} \int_{y_1}^{\infty} (\varpi'^2 + k^2\varpi^2)\, Y^{-1} dy \tag{4.2a}$$

$$= |k\varpi_1|^{-2} \int_{y_1}^{\infty} (|\varpi'|^2 + k^2\, |\varpi|^2)\, Y^{-1} dy. \tag{4.2b}$$

The integrals (4.1*a*) and (4.2*a*) are (*a*) stationary with respect to first-order variations of θ and ϖ about the true solutions of (2.4) and (2.5), respectively, provided that the approximate solutions are differentiable and vanish uniformly as $ky \to \infty$, (*b*) absolute minima with respect to arbitrary variations about the true solutions, and (*c*) invariant under scale transformations of θ and ϖ. These properties render (4.1*a*) and (4.2*a*) quite suitable for the approximate determination of Ω_1, bounding it from above and below.

The integrals (4.1*b*) and (4.2*b*) are identical with (4.1*a*) and (4.2*a*) for real θ and ϖ, but we have included them by virtue of their utility in the more general case of unrestricted, complex c.

Perhaps the simplest approximations in connexion with the variational integrals of (4.1) and (4.2) are the solutions to (2.4) and (2.5) for $U'(y) \equiv 0$, namely

$$\theta = e^{-ky} \quad \text{and} \quad \varpi = e^{-ky}. \tag{4.3a,b}$$

Designating the corresponding approximations to Ω_1 by Ω_{1-} and Ω_{1+} (implying lower and upper bounds, respectively), we obtain

$$1/\Omega_{1-} = 2k^2 \int_{y_1}^{\infty} e^{-2k(y-y_1)}\, Y\, dy \tag{4.3}$$

and

$$\Omega_{1+} = 2 \int_{y_1}^{\infty} e^{-2k(y-y_1)}\, Y^{-1} dy. \tag{4.4}$$

We remark that the approximation (4.4) also may be obtained by substituting $\Omega = (1/kY) + f(y)$ in (2.7), neglecting f^2, and integrating the result between

† It is readily seen (for example, by comparing (4.1*b*) to (4.1*a*)) that these assumptions render θ/θ_1 and ϖ/ϖ_1 real and hence that θ and ϖ are real within constant, possibly complex factors. This remark justifies the subsequent application of Sturm's oscillation theorem (or a trivial generalization thereof).

$y = y_1$ and $y = \infty$. We also may show that (4.4) is an asymptotic approximation to Ω_1 as $ky_1 \to \infty$; however, we intend to use (4.4) for small values of ky_1.

5. Application to turbulent boundary layer

We now apply the foregoing development to the mean velocity profile for a turbulent boundary layer, the results being required in connexion with the afore-mentioned surface-wave problem (Miles 1962). An approximation to this profile that agrees well with observations is given by (Miles 1957*b*)

$$U = (U_*^2/\nu)\,y \quad (0 \leqslant y \leqslant y_1), \tag{5.1}$$

in the laminar sublayer and by

$$U = U_1 + (U_*/\kappa)\,[\alpha - \tanh(\tfrac{1}{2}\alpha)] \tag{5.2a}$$

and
$$y = y_1 + (\nu/2\kappa U_*)\sinh\alpha \tag{5.2b}$$

outside this sublayer, where ν is the kinematic viscosity, κ is von Kármán's constant,
$$U_1 = (U_*^2/\nu)\,y_1 \tag{5.3}$$

is the velocity at the edge of the laminar sublayer, and α is a parametric variable. The asymptotic form of (5.2) is

$$U \sim U_1 + \frac{U_*}{\kappa}\left[\log\left(\frac{4\kappa U_* y}{\nu}\right) - 1 + O\left(\frac{y_1}{y}\right)\right], \tag{5.4}$$

which relates the usual additive constant in the logarithmic profile to $\kappa U_1/U_*$. We shall identify y_1 in this representation with y_1 in (1.7), thereby implying $ky_1 \ll 1$.

We may represent our results as functions of the parameters

$$R = \kappa U_*/k\nu \tag{5.5}$$

and
$$A = \kappa(U_1 - c)/U_*. \tag{5.6}$$

Introducing the change of variable

$$\Omega(y) = (\kappa^2/kU_*^2)\,G(\alpha) \tag{5.7}$$

in (2.7), we then obtain

$$2RG'(\alpha) = [v^2(\alpha)\,G^2(\alpha) - v^{-2}(\alpha)]\cosh\alpha, \tag{5.8a}$$

where
$$v(\alpha) = A + \alpha - \tanh(\tfrac{1}{2}\alpha). \tag{5.8b}$$

Substituting (5.1) and (5.7) into (3.4) and (3.5), we may place the result in the form
$$w = (\kappa c/U_*)\,W(R, A), \tag{5.9}$$

where
$$W(R, A) = RG(0) - A^{-1}. \tag{5.10}$$

Turning to the variational approximations (4.3) and (4.4), substituting (5.2*a*, *b*) therein and then determining the corresponding approximations to W through (5.7) and (5.10) yields

$$W_- = -A^{-1} + R^2\left[\int_0^\infty \exp(-R^{-1}\sinh\alpha)\,v^2(\alpha)\cosh\alpha\,d\alpha\right]^{-1} \tag{5.11}$$

and
$$W_+ = -A^{-1} + \int_0^\infty \exp(-R^{-1}\sinh\alpha)\,v^{-2}(\alpha)\cosh\alpha\,d\alpha. \tag{5.12}$$

Numerical results for W_-, W_+ and $\frac{1}{2}(W_- + W_+)$ are compared with those obtained through the direct integration of (5.8) in table 1 for $\kappa = 0{\cdot}4$. We conclude that the individual approximations W_- and W_+ are likely to be adequate only for rather large values of the Reynolds number R, but that the mean variational approximation $\frac{1}{2}(W_- + W_+)$ should be adequate for $R \geqslant 10$ (smaller values of R were of no interest in the contemplated application of the results) and $A \geqslant 0{\cdot}4$

A	R	W_+	W_-	$\frac{1}{2}(W_+ + W_-)$	W
0·4	10	2·827	−0·6185	1·104	1·272
	10^2	8·676	2·818	5·747	5·875
	10^3	30·64	20·78	25·71	25·88
	10^4	147·1	126·0	136·6	136·9
1	10	1·016	0·2144	0·630	0·6634
	10^2	5·215	3·137	4·176	4·240
	10^3	23·40	18·60	21·00	21·11
	10^4	125·1	111·8	118·5	118·8
2	10	0·3678	0·1803	0·2741	0·2850
	10^2	3·120	2·380	2·750	2·775
	10^3	16·93	14·63	15·78	15·84
	10^4	99·72	91·85	95·79	95·96
4	10	0·0777	0·0473	0·0625	0·0640
	10^2	1·555	1·368	1·462	1·467
	10^3	10·34	9·533	9·937	9·956
	10^4	68·37	64·86	66·62	66·68

TABLE 1. A comparison of the mean variational approximation $\frac{1}{2}(W_+ + W_-)$, as determined from (5.11) and (5.12), with the result for W given by (5.10) after numerical integration of (5.8).

($U_1 - c \geqslant U_*$). Granted the availability of a high-speed computer, the integration of (5.8)—or, in the general case, of (2.7)—is straightforward, but even then the use of the mean variational approximation may be more economical.

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